

# IRREDUCIBILITY OF THE MONODROMY REPRESENTATION OF LAURICELLA'S $F_C$

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ABSTRACT. Let  $E_C$  be the hypergeometric system of differential equations satisfied by Lauricella's hypergeometric series  $F_C$  of  $m$  variables. We show that the monodromy representation of  $E_C$  is irreducible under our assumption consisting of  $2^{m+1}$  conditions for parameters. We also show that the monodromy representation is reducible if one of them is not satisfied.

## 1. INTRODUCTION

Lauricella's hypergeometric series  $F_C$  of  $m$  variables  $x_1, \dots, x_m$  with complex parameters  $a, b, c_1, \dots, c_m$  is defined by

$$F_C(a, b, c; x) = \sum_{n_1, \dots, n_m=0}^{\infty} \frac{(a, n_1 + \dots + n_m)(b, n_1 + \dots + n_m)}{(c_1, n_1) \dots (c_m, n_m) n_1! \dots n_m!} x_1^{n_1} \dots x_m^{n_m},$$

where  $x = (x_1, \dots, x_m)$ ,  $c = (c_1, \dots, c_m)$ ,  $c_1, \dots, c_m \notin \{0, -1, -2, \dots\}$ , and  $(c_1, n_1) = \Gamma(c_1 + n_1)/\Gamma(c_1)$ . This series converges in the domain

$$D_C = \left\{ (x_1, \dots, x_m) \in \mathbb{C}^m \mid \sum_{k=1}^m \sqrt{|x_k|} < 1 \right\}.$$

It is shown in [3] that the hypergeometric system  $E_C = E_C(a, b, c)$  of differential equations satisfied by  $F_C(a, b, c; x)$  is a holonomic system of rank  $2^m$  with the singular locus

$$S = \left( \prod_{k=1}^m x_k \cdot R(x) = 0 \right) \subset \mathbb{C}^m,$$

$$R(x_1, \dots, x_m) = \prod_{\varepsilon_1, \dots, \varepsilon_m = \pm 1} \left( 1 + \sum_{k=1}^m \varepsilon_k \sqrt{x_k} \right),$$

and that the system  $E_C$  is irreducible (in the sense of  $D$ -modules) if and only if

$$(1) \quad a - \sum_{k=1}^m i_k c_k, \quad b - \sum_{k=1}^m i_k c_k \notin \mathbb{Z}, \quad \forall I = (i_1, \dots, i_m) \in \{0, 1\}^m.$$

It is classically known that there are  $2^m$  solutions to  $E_C(a, b, c)$  expressed in terms of  $F_C$  with different parameters(see (3)). If parameters satisfy (1) and  $c_1, \dots, c_m \notin$

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$\mathbb{Z}$  then they form a fundamental system of solutions to  $E_C(a, b, c)$  in a simply connected domain in  $D_C - S$ .

Let  $X$  be the complement of the singular locus  $S$ . The fundamental group of  $X$  is generated by  $m + 1$  loops  $\rho_0, \rho_1, \dots, \rho_m$  (see §2.2). In [2], we express the circuit transformations  $\mathcal{M}_i$  along  $\rho_i$  ( $i = 0, \dots, m$ ) by using the  $2^m$  solutions and the intersection form on twisted homology groups associated with Euler-type integrals of solutions to  $E_C$ . These expressions are independent of the choice of a basis of the twisted homology group. The circuit transformations  $\mathcal{M}_i$  are also studied in [6] by the specification of the intersection form regarded as indeterminate.

In this paper, we show the following.

**Theorem 1.1** (Main theorem). *The monodromy representation*

$$\mathcal{M} : \pi_1(X, \dot{x}) \rightarrow GL(\text{Sol}_{\dot{x}})$$

*is irreducible under the assumption (1), where  $\text{Sol}_{\dot{x}} = \text{Sol}_{\dot{x}}(a, b, c)$  is the local solution space to  $E_C(a, b, c)$  around a point  $\dot{x} \in D_C - S$ .*

We also show that the monodromy representation is reducible if one of the assumption (1) is not satisfied. We specify an invariant subspace in  $\text{Sol}_{\dot{x}}$  under the monodromy representation  $\mathcal{M}$  in this case.

Note that under the assumption (1), for example,  $c_k$  may be an integer. In such a case, the solutions (3) expressed by  $F_C$  do not form a basis of the local solution space. We give a linear transformation of them so that the transformed solutions are valid even in cases where any of  $c_k$ 's are integers. We construct it inductively on  $m$  using tensor products of matrices.

We remark that the irreducibility of the monodromy representation  $\mathcal{M}$  is implied from that of the system  $E_C$  under the assumption (1). We prove it explicitly by using properties of the circuit transformations in [2], not applying results of  $D$ -modules. We here briefly explain our idea of the proof of the main theorem. It is shown in [2] that the 1-eigenspace  $V$  of  $\mathcal{M}_0$  is  $(2^m - 1)$ -dimensional. Let  $f_0 \in \text{Sol}_{\dot{x}}$  (corresponding to  $e_{1, \dots, 1}$  in §3.2) be a non-zero vector in its orthogonal complement with respect to the intersection form. It is quite easy to give a basis of the whole space  $\text{Sol}_{\dot{x}}$  by actions  $\mathcal{M}_1, \dots, \mathcal{M}_m$  on  $f_0$ . Let  $W$  be an invariant subspace of  $\text{Sol}_{\dot{x}}$  under the monodromy representation  $\mathcal{M}$ . If  $W \not\subset V$  then we can show  $f_0 \in W$ , which yields that  $W$  becomes the whole space  $\text{Sol}_{\dot{x}}$  by the previous fact. Otherwise, we can show that  $W$  becomes the zero space by the perfectness of the intersection form.

## 2. PRELIMINARIES

Except in §5, we assume the conditions for parameters  $a, b, c_1, \dots, c_m$  in (1) (it is equivalent to (2) or (4) mentioned below).

In this section, we collect some facts about Lauricella's  $F_C$  mentioned in [1], [2], [3] and [5].

*Notation 2.1.* We put

$$\alpha = \exp(2\pi\sqrt{-1}a), \quad \beta = \exp(2\pi\sqrt{-1}b), \quad \gamma_k = \exp(2\pi\sqrt{-1}c_k) \quad (k = 1, \dots, m).$$

We often regard  $\alpha, \beta$  and  $\gamma_k$  as indeterminants, and consider the rational function field  $\mathbb{C}(\alpha, \beta, \gamma) = \mathbb{C}(\alpha, \beta, \gamma_1, \dots, \gamma_m)$ . For a rational function  $g(\alpha, \beta, \gamma_1, \dots, \gamma_m) \in \mathbb{C}(\alpha, \beta, \gamma)$ , we denote  $g(\alpha, \beta, \gamma_1, \dots, \gamma_m)^\vee = g(\alpha^{-1}, \beta^{-1}, \gamma_1^{-1}, \dots, \gamma_m^{-1})$ .

Under these notations, the condition (1) is equivalent to

$$(2) \quad \alpha - \prod_{k=1}^m \gamma_k^{i_k}, \quad \beta - \prod_{k=1}^m \gamma_k^{i_k} \neq 0, \quad \forall I = (i_1, \dots, i_m) \in \{0, 1\}^m.$$

For example,  $\gamma_k = 1$  or  $\alpha\beta - (-1)^{m-1} \prod_{k=1}^m \gamma_k = 0$  are allowed.

Note that though in [2] the indices  $I$  run the subsets of  $\{1, \dots, m\}$ , in this paper we use  $\{0, 1\}^m$  as a set of indices. The correspondence is given by

$$\{1, \dots, m\} \supset \{i_1, \dots, i_r\} \longleftrightarrow e_{i_1} + \dots + e_{i_r} \in \{0, 1\}^m,$$

where  $e_k$  is the  $k$ -th unit vector of size  $m$ . We put  $|I| = \sum_{k=1}^m i_k$ .

**2.1. System of differential equations.** Let  $\partial_k$  ( $k = 1, \dots, m$ ) be the partial differential operator with respect to  $x_k$ . We set  $\theta_k = x_k \partial_k$ ,  $\theta = \sum_{k=1}^m \theta_k$ . Lauricella's  $F_C(a, b, c; x)$  satisfies differential equations

$$[\theta_k(\theta_k + c_k - 1) - x_k(\theta + a)(\theta + b)] f(x) = 0, \quad k = 1, \dots, m.$$

The system generated by them is called Lauricella's hypergeometric system  $E_C(a, b, c)$  of differential equations. The system  $E_C(a, b, c)$  is a holonomic system of rank  $2^m$  with the singular locus  $S$ . It is shown in [3] that the system  $E_C(a, b, c)$  is irreducible, that is, the system  $E_C(a, b, c)$  defines a maximal ideal in the ring of differential operators with rational function coefficients, if and only if the parameters  $a, b, c_1, \dots, c_m$  satisfy (1).

For an element  $I = (i_1, \dots, i_m)$  of  $\{0, 1\}^m$ , we set

$$(3) \quad F_I(x) = \frac{\prod_{k=1}^m \Gamma((-1)^{i_k}(1 - c_k))}{\Gamma(1 - a^I)\Gamma(1 - b^I)} \cdot \prod_{k=1}^m x_k^{i_k(1 - c_k)} \cdot F_C(a^I, b^I, c^I; x),$$

where

$$\begin{aligned} a^I &= a + \sum_{k=1}^m i_k(1 - c_k), & b^I &= b + \sum_{k=1}^m i_k(1 - c_k), \\ c^I &= (c_1 + 2i_1(1 - c_1), \dots, c_m + 2i_m(1 - c_m)). \end{aligned}$$

Note that the assumption (1) is equivalent to

$$(4) \quad a^I, b^I \notin \mathbb{Z}, \quad \forall I = (i_1, \dots, i_m) \in \{0, 1\}^m,$$

and that

$$c_k + 2i_k(1 - c_k) = \begin{cases} c_k & \text{if } i_k = 0, \\ 2 - c_k & \text{if } i_k = 1. \end{cases}$$

*Example 2.2.* We give examples for  $m = 1$  (we put  $c_1 = c$ ,  $x_1 = x$ ):

$$\begin{aligned}
F_0(x) &= \frac{\Gamma(1-c)}{\Gamma(1-a)\Gamma(1-b)} F_C(a, b, c; x) \\
&= \frac{\Gamma(1-c)\Gamma(c)}{\Gamma(1-a)\Gamma(1-b)\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)\Gamma(1+n)} x^n \\
&= \frac{\sin(\pi a)\sin(\pi b)}{\pi \sin(\pi c)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)\Gamma(1+n)} x^n, \\
F_1(x) &= \frac{\Gamma(c-1)}{\Gamma(c-a)\Gamma(c-b)} x^{1-c} F_C(a+1-c, b+1-c, 2-c; x) \\
&= \frac{\Gamma(c-1)\Gamma(1-(c-1))}{\Gamma(c-a)\Gamma(c-b)\Gamma(1-c+a)\Gamma(1-c+b)} \sum_{n=0}^{\infty} \frac{\Gamma(a-c+1+n)\Gamma(b-c+1+n)}{\Gamma(2-c+n)\Gamma(1+n)} x^{n+1-c}, \\
&= -\frac{\sin(\pi(a-c))\sin(\pi(b-c))}{\pi \sin(\pi c)} \sum_{n'=1-c}^{\infty} \frac{\Gamma(a+n')\Gamma(b+n')}{\Gamma(c+n')\Gamma(1+n')} x^{n'},
\end{aligned}$$

where  $n' = n + 1 - c$  and  $\sum_{n'=1-c}^{\infty}$  means the sum of  $n'$  running over the set  $1 - c + \mathbb{N} = \{1 - c + n \mid n \in \mathbb{N}\}$ .

The functions  $\{F_I(x)\}_{I \in \{0,1\}^m}$  form a basis of the local solution space  $Sol_{\dot{x}} = Sol_{\dot{x}}(a, b, c)$  to the system  $E_C(a, b, c)$  around a point  $\dot{x}$  in  $D_C - S$  under conditions (1) and

$$(5) \quad c_1, \dots, c_m \notin \mathbb{Z}.$$

We set a (row) vector valued function

$$\mathbf{F}(x) = (\dots, F_I(x), \dots),$$

where  $I \in \{0, 1\}^m$  are aligned by the pure lexicographic order as

$$(0, \dots, 0), (1, 0, \dots, 0), (0, 1, \dots, 0), (1, 1, \dots, 0), (0, 0, 1, \dots, 0), \dots, (1, \dots, 1).$$

Note that its entries have factors

$$1, x_1^{1-c_1}, x_2^{1-c_2}, x_1^{1-c_1} x_2^{1-c_2}, x_3^{1-c_3}, \dots, x_1^{1-c_1} x_2^{1-c_2} \dots x_m^{1-c_m},$$

respectively.

**2.2. Monodromy representation.** Put  $\dot{x} = (\frac{1}{2m^2}, \dots, \frac{1}{2m^2}) \in X$ . For  $\rho \in \pi_1(X, \dot{x})$  and  $g \in Sol_{\dot{x}}$ , let  $\rho_* g$  be the analytic continuation of  $g$  along  $\rho$ . Since  $\rho_* g$  is also a solution to  $E_C(a, b, c)$ , the map  $\rho_* : Sol_{\dot{x}} \rightarrow Sol_{\dot{x}}$ ;  $g \mapsto \rho_* g$  is a  $\mathbb{C}$ -linear automorphism which satisfies  $(\rho \cdot \rho')_* = \rho'_* \circ \rho_*$  for  $\rho, \rho' \in \pi_1(X, \dot{x})$ . Here, the composition  $\rho \cdot \rho'$  of loops  $\rho$  and  $\rho'$  is defined as the loop going first along  $\rho$ , and then along  $\rho'$ . We thus obtain a representation

$$\mathcal{M} : \pi_1(X, \dot{x}) \rightarrow GL(Sol_{\dot{x}})$$

of  $\pi_1(X, \dot{x})$ , where  $GL(V)$  is the general linear group on a  $\mathbb{C}$ -vector space  $V$ . This representation  $\mathcal{M}$  is called the monodromy representation of  $E_C(a, b, c)$ .

Let  $\rho_0, \rho_1, \dots, \rho_m$  be loops in  $X$  so that

- $\rho_0$  turns the hypersurface  $(R(x) = 0)$  around the point  $(\frac{1}{m^2}, \dots, \frac{1}{m^2})$ , positively,
- $\rho_k$  ( $k = 1, \dots, m$ ) turns the hyperplane  $(x_k = 0)$ , positively.

For explicit definitions of them, see [2].

**Fact 2.3** ([2]). *The loops  $\rho_0, \rho_1, \dots, \rho_m$  generate the fundamental group  $\pi_1(X, \dot{x})$ . Moreover, if  $m \geq 2$ , then they satisfy the following relations:*

$$\rho_i \rho_j = \rho_j \rho_i \quad (i, j = 1, \dots, m), \quad (\rho_0 \rho_k)^2 = (\rho_k \rho_0)^2 \quad (k = 1, \dots, m).$$

In [2],  $m + 1$  linear maps  $\mathcal{M}_i = \mathcal{M}(\rho_i)$  ( $i = 0, \dots, m$ ) are investigated in terms of twisted homology groups and the intersection form. In this paper, we do not explain them. What we need is the following fact.

**Fact 2.4** ([1]). *Suppose (1) and (5).*

- (i) *By integration, the twisted homology group is isomorphic to the solution space  $Sol_{\dot{x}}$ .*
- (ii) *We can construct twisted cycles  $\{\Delta_I\}_I$  that correspond to  $\{F_I\}_I$ .*
- (iii) *The intersection matrix  $H = (H_{I,I'})_{I,I'}$  with respect to the basis  $\{\Delta_I\}_I$  is diagonal, and its  $(I, I)$ -entry is*

$$H_{I,I} = \prod_{k=1}^m \frac{(-1)^{i_k} \gamma_k^{1-i_k}}{\gamma_k - 1} \cdot \frac{(\alpha - \prod_{k=1}^m \gamma_k^{i_k})(\beta - \prod_{k=1}^m \gamma_k^{i_k})}{(\alpha - \prod_{k=1}^m \gamma_k)(\beta - 1)}.$$

By using this fact and properties of the intersection form, we can induce the intersection form on  $Sol_{\dot{x}}$ .

**Definition 2.5.** We assume (1) and (5). We define a bilinear form (called the intersection form)

$$\mathcal{I}(\cdot, \cdot) : Sol_{\dot{x}} \times Sol_{\dot{x}} \rightarrow \mathbb{C}(\alpha, \beta, \gamma)$$

as follows. For any  $F(x), G(x) \in Sol_{\dot{x}}$ , we express them as linear combinations of the basis  $\{F_I\}_I$ :

$$F(x) = \mathbf{F}(x) \cdot \mathbf{f}, \quad G(x) = \mathbf{F}(x) \cdot \mathbf{g}, \quad \mathbf{f}, \mathbf{g} \in \mathbb{C}(\alpha, \beta, \gamma)^{2^m},$$

and define

$$\mathcal{I}(F(x), G(x)) = {}^t \mathbf{f} \cdot H \cdot \mathbf{g}^\vee.$$

*Remark 2.6.* Let  $\mathcal{I}_H$  be the intersection form on  $Sol_{\dot{x}}$  induced from that on the twisted homology group by the isomorphism in Fact 2.4 (i). The intersection form  $\mathcal{I}$  coincides with  $\mathcal{I}_H$  modulo a constant multiple which never vanishes under the conditions (1) and (5).

**Corollary 2.7.** *Under the conditions (1) and (5), the intersection form  $\mathcal{I}$  is a monodromy invariant form, that is, for any loops  $\rho \in \pi_1(X, \dot{x})$ , we have*

$$\mathcal{I}(\mathcal{M}(\rho)(F(x)), \mathcal{M}(\rho)(G(x))) = \mathcal{I}(F(x), G(x)).$$

*In other words,  $H$  satisfies*

$${}^t M_\rho \cdot H \cdot M_\rho^\vee = H,$$

*where  $M_\rho$  is the representation matrix of  $\mathcal{M}(\rho)$  with respect to the basis  $\{F_I\}_I$ .*

Let  $M_i$  be the representation matrix of  $\mathcal{M}_i$  ( $i = 0, \dots, m$ ) with respect to the basis  $\{F_I\}_I$ . We give explicit expressions of them.

**Fact 2.8** ([2]). We assume (1), (5) and  $\lambda = (-1)^{m-1}\alpha^{-1}\beta^{-1}\prod_{k=1}^m\gamma_k \neq 1$ . For  $k = 1, \dots, m$ , the representation matrix  $M_k$  is diagonal, and its  $(I, I)$ -entry is  $\gamma_k^{-i_k}$ . The representation matrix  $M_0$  is expressed as

$$M_0 = E_{2^m} - \frac{1 - \lambda}{\mathbf{1} \cdot H \cdot \mathbf{1}} \cdot \mathbf{1} \cdot {}^t\mathbf{1} \cdot H = E_{2^m} - \frac{(\beta - 1)(\alpha - \prod_{k=1}^m \gamma_k)}{\alpha\beta} \cdot \mathbf{1} \cdot {}^t\mathbf{1} \cdot H,$$

where  $E_{2^m}$  is the unit matrix of size  $2^m$ ,  $\mathbf{1}$  is the column vector of size  $2^m$  with all entries 1, and  $H$  is the intersection matrix given in Fact 2.4.

*Remark 2.9.* These expressions are obtained from consideration to eigenvectors of each  $\mathcal{M}_i \in GL(\text{Sol}_{\dot{x}})$ .

- (i)  $\mathcal{M}_k$  ( $k = 1, \dots, m$ );  $F_I$  is an eigenvector of eigenvalue  $\gamma_k^{-1}$  (resp. 1) if  $i_k = 1$  (resp.  $i_k = 0$ ), where  $I = (i_1, \dots, i_m)$ .
- (ii)  $\mathcal{M}_0$ ; the eigenvalues of  $\mathcal{M}_0$  are  $\lambda$  and 1. The eigenspace of eigenvalue  $\lambda$  is one-dimensional and spanned by

$$f_0 = \sum_{I \in \{0,1\}^m} F_I,$$

which corresponds to  $\mathbf{1}$  when we take the basis  $\{F_I\}_I$ . The eigenspace of  $\mathcal{M}_0$  of eigenvalue 1 is characterized as  $\{g \in \text{Sol}_{\dot{x}} \mid \mathcal{I}(g, f_0) = 0\}$ .

- (iii) The first expression of  $M_0$  is stable under the non-zero scalar multiple to  $H$ .

### 3. ANOTHER BASIS

In fact,  $\{F_I\}$  does not form a basis of  $\text{Sol}_{\dot{x}}$  when  $c_i$ 's are integers. In this section, we introduce another basis  $\{\tilde{F}_I\}_I$  which is a well-defined basis even if  $c_i$ 's are integers, and we give the circuit matrices with respect to this basis. Note that these do not coincide with solutions obtained by integrating the twisted cycles defined in [2, §6]

**3.1. Basis of  $\text{Sol}_{\dot{x}}$ .** First, we construct a basis of  $\text{Sol}_{\dot{x}}$ .

**Lemma 3.1.** Let  $I = (i_1, i_2, \dots, i_m)$  be any element of  $\{0, 1\}^m$  and  $p = (p_1, p_2, \dots, p_m)$  be any element of  $\mathbb{Z}^m$ . Then the limit function

$$\lim_{c \rightarrow p} (\gamma_1 - 1)(\gamma_2 - 1) \cdots (\gamma_m - 1) F_I(x)$$

is well-defined and not identically zero.

*Proof.* We have only to note that  $F_I(x)$  has the factor

$$\prod_{k=1}^m \Gamma(c_k - i_k) \Gamma(1 - (c_k - i_k)) = \prod_{k=1}^m \frac{(-1)^{i_k} \pi}{\sin(\pi c_k)},$$

which cancels out  $(\gamma_1 - 1)(\gamma_2 - 1) \cdots (\gamma_m - 1)$ . □

**Lemma 3.2.** Let  $I = (i_1, \dots, \overset{k\text{-th}}{0}, \dots, i_m)$  and  $I' = (i_1, \dots, \overset{k\text{-th}}{1}, \dots, i_m)$  be elements of  $\{0, 1\}^m$  and  $p_k$  be an integer. Then the limit function

$$\lim_{c_k \rightarrow p_k} (\gamma_m - 1) \cdots (\gamma_{k+1} - 1)(\gamma_{k-1} - 1) \cdots (\gamma_1 - 1)(F_I(x) + F_{I'}(x))$$

is well-defined and not identically zero.

*Proof.* We show the case  $m = 1$  (we put  $p_1 = p$ ). Firstly, we assume  $p = 1$ . By Example 2.2,

$$F_0(x) = \frac{s_0}{\sin(\pi c)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)\Gamma(1+n)} x^n,$$

$$F_1(x) = -\frac{s_1(c)}{\sin(\pi c)} \sum_{n'=1-c}^{\infty} \frac{\Gamma(a+n')\Gamma(b+n')}{\Gamma(c+n')\Gamma(1+n')} x^{n'},$$

where  $n' = n + 1 - c$ ,  $s_0 = \frac{\sin(\pi a)\sin(\pi b)}{\pi}$ ,  $s_1(c) = \frac{\sin(\pi(a-c))\sin(\pi(b-c))}{\pi}$ . Both functions  $\sin(\pi c)F_0(x)$  and  $-\sin(\pi c)F_1(x)$  converge to

$$\sum_{n=0}^{\infty} s_0 \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(1+n)\Gamma(1+n)} x^n = \sum_{n=0}^{\infty} A_n x^n$$

as  $c \rightarrow p = 1$ . Apply l'Hôpital's rule to the function

$$F_0(x) + F_1(x) = \frac{1}{\sin(\pi c)} \cdot \left[ \sum_{n=0}^{\infty} \left( s_0 \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)\Gamma(1+n)} - A_n \right) x^n \right. \\ \left. - \sum_{n'=1-c}^{\infty} \left\{ \left( s_1(c) \frac{\Gamma(a+n')\Gamma(b+n')}{\Gamma(c+n')\Gamma(1+n')} - A_n \right) x^{n'} - A_n (x^n - x^{n'}) \right\} \right]$$

to verify that its limit as  $c \rightarrow p = 1$  exists, where  $n = n' + c - 1$  in the second sum.

Note that  $\lim_{c \rightarrow 1} \frac{A_n(x^n - x^{n'})}{\sin(\pi c)}$  yields the factor  $\log x$ .

Secondly, we assume  $p \geq 2$ . In this case, the sum  $\sum_{n'=1-c}^{\infty}$  has negative terms for  $n' = 1 - p, \dots, -1$  as  $c \rightarrow p$ . Since  $\lim_{c \rightarrow p} \frac{1}{\sin(\pi c)} \cdot \frac{1}{\Gamma(n' + 1)}$  converges to a non-zero value, these negative terms are well-defined. Thus  $F_0(x) + F_1(x)$  consists of these finite terms and the infinite sum considered in the case  $p = 1$ .

Thirdly, we assume  $p \leq 0$ . By regarding  $n$  as  $n + p$  in the sums of  $F_0(x)$  and  $F_1(x)$ , we can show that  $F_0(x) + F_1(x)$  is well-defined and not identically zero as in the previous consideration.

For a general  $m$ , use a similar argument by regrading the variables except  $x_k$  as constants. Note that the limit function has the factor  $\log x_k$  coming from

$$\lim_{n_k \rightarrow n'_k} \frac{x_k^{n_k} - x_k^{n'_k}}{n_k - n'_k}.$$

□

We define the tensor product  $A \otimes B$  of matrices  $A$  and  $B = (b_{ij})_{\substack{1 \leq i \leq r \\ 1 \leq j \leq s}}$  as

$$A \otimes B = \begin{pmatrix} A b_{11} & A b_{12} & \cdots & A b_{1s} \\ A b_{21} & A b_{22} & \cdots & A b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ A b_{r1} & A b_{r2} & \cdots & A b_{rs} \end{pmatrix}.$$

We remark that this is different from the usual definition. We fix the number  $m$  of variables. We set

$$G_k = \begin{pmatrix} 1 & 0 \\ 0 & \gamma_k^{-1} \end{pmatrix}, \quad Q_k = \begin{pmatrix} 1 - \gamma_k & 1 \\ 0 & 1 \end{pmatrix},$$

for  $k = 1, \dots, m$  and

$$P_m = Q_1 \otimes Q_2 \otimes \cdots \otimes Q_m.$$

By using these notations, the matrices  $M_1, \dots, M_m$  given in Fact 2.8 is expressed as

$$M_k = E_2 \otimes \cdots \otimes E_2 \otimes \underset{k\text{-th}}{G_k} \otimes E_2 \otimes \cdots \otimes E_2 \quad (k = 1, \dots, m)$$

where  $E_2$  is the unit matrix of size 2. For example, we have

$$M_1 = G_1 = \begin{pmatrix} 1 & 0 \\ 0 & \gamma_1^{-1} \end{pmatrix}, \quad P_1 = Q_1 = \begin{pmatrix} 1 - \gamma_1 & 1 \\ 0 & 1 \end{pmatrix}$$

in the case  $m = 1$ , and

$$M_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \gamma_1^{-1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \gamma_1^{-1} \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma_2^{-1} & 0 \\ 0 & 0 & 0 & \gamma_2^{-1} \end{pmatrix},$$

$$P_2 = Q_1 \otimes Q_2 = \begin{pmatrix} (1 - \gamma_1)(1 - \gamma_2) & 1 - \gamma_2 & 1 - \gamma_1 & 1 \\ 0 & 1 - \gamma_2 & 0 & 1 \\ 0 & 0 & 1 - \gamma_1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

in the case  $m = 2$ . Note that

$$P_m = \begin{pmatrix} P_{m-1}(1 - \gamma_m) & P_{m-1} \\ O & P_{m-1} \end{pmatrix},$$

where  $O$  is the square zero matrix of size  $2^{m-1}$ . We have

$$(6) \quad \det(P_m) = \prod_{k=1}^m (1 - \gamma_k)^{2^{m-1}},$$

since  $\det(P_1) = 1 - \gamma_1$  and

$$\det(P_m) = \det((1 - \gamma_m)P_{m-1}) \det(P_{m-1}) = (1 - \gamma_m)^{2^{m-1}} \det(P_{m-1})^2.$$

We use a new basis given by

$$\tilde{\mathbf{F}}(x) = (\dots, \tilde{F}_I(x), \dots) = \mathbf{F}(x) \cdot P_m.$$

The vector-valued function  $\tilde{\mathbf{F}}(x)$  takes the form

$$\left( (1 - \gamma_1)F_0(x), F_0(x) + F_1(x) \right)$$

for  $m = 1$ , and the form

$$\begin{aligned} & \left( (1 - \gamma_1)(1 - \gamma_2)F_{00}(x), (1 - \gamma_2)(F_{00}(x) + F_{10}(x)), \right. \\ & \left. (1 - \gamma_1)(F_{00}(x) + F_{01}(x)), F_{00}(x) + F_{10}(x) + F_{01}(x) + F_{11}(x) \right) \end{aligned}$$

for  $m = 2$ .

**Theorem 3.3.** *The vector-valued function  $\tilde{\mathbf{F}}(x)$  gives a basis of the space  $\text{Sol}_{\hat{x}}$  of the local solutions to  $E_C(a, b, c)$  around  $\hat{x}$  even in cases  $c_k \in \mathbb{Z}$  ( $k = 1, \dots, m$ ).*



*Proof.* The entries of  $\tilde{\mathbf{F}}(x)$  consist of

$$\begin{aligned} & \left[ \prod_{k=1}^m (1 - \gamma_k) \right] F_{0\dots 0}, \\ & \left[ \prod_{\substack{k \neq i \\ 1 \leq k \leq m}} (1 - \gamma_k) \right] (F_{0\dots 0} + F_{e_i}), \\ & \left[ \prod_{\substack{k \neq i,j \\ 1 \leq k \leq m}} (1 - \gamma_k) \right] (F_{0\dots 0} + F_{e_i} + F_{e_j} + F_{e_i+e_j}), \\ & \vdots \\ & (1 - \gamma_k) \sum_{I \in \{0,1\}^m} (1 - i_k) F_I, \\ & \sum_{I \in \{0,1\}^m} F_I, \end{aligned}$$

where  $e_k$  is the  $k$ -th unit vector of size  $m$  and  $I = (i_1, \dots, i_m)$ . The functions in the first and second lines are well-defined by Lemmas 3.1 and 3.2. Since the functions

$$\begin{aligned} & \left[ \prod_{\substack{k \neq i \\ 1 \leq k \leq m}} (1 - \gamma_k) \right] [(F_{0\dots 0} + F_{e_i}) + (F_{e_j} + F_{e_i+e_j})], \\ & \left[ \prod_{\substack{k \neq j \\ 1 \leq k \leq m}} (1 - \gamma_k) \right] [(F_{0\dots 0} + F_{e_j}) + (F_{e_i} + F_{e_i+e_j})] \end{aligned}$$

are well-defined by Lemma 3.2, the function

$$\left[ \prod_{\substack{k \neq i,j \\ 1 \leq k \leq m}} (1 - \gamma_k) \right] (F_{0\dots 0} + F_{e_i} + F_{e_j} + F_{e_i+e_j})$$

is also well-defined. In this way, we can show that the entries of  $\tilde{\mathbf{F}}(x)$  are well-defined even in cases  $c_k \in \mathbb{Z}$ . In the case  $c_k \in \mathbb{Z}$ , the functions  $\tilde{F}_I(x)$  has the factor  $\log x_k$ , if  $i_k = 1$ . This implies that  $\tilde{F}_I(x)$ 's are also linearly independent in such a case.  $\square$

**3.2. Representation matrices and the intersection matrix.** Next, we consider the representation matrices of  $\mathcal{M}_i$ 's and the intersection matrix with respect to the new basis  $\{\tilde{F}_I\}_I$ .

In the below discussion, we often use the following equality which is shown by a straightforward calculation:

$$\begin{aligned} (7) \quad & \sum_{J \leq I} (-1)^{|J|} \left( \alpha \beta \prod_{k=1}^m \gamma_k^{1-j_k} + \prod_{k=1}^m \gamma_k^{1+j_k} \right) \\ & = \left( \alpha \beta + (-1)^{|I|} \prod_{k=1}^m \gamma_k^{i_k} \right) \prod_{k=1}^m (\gamma_k^{1-i_k} (\gamma_k - 1)^{i_k}), \end{aligned}$$

where  $I = (i_1, \dots, i_m)$ ,  $J = (j_1, \dots, j_m)$  and we define a partial order  $\leq$  on  $\{0, 1\}^m$  by

$$J \leq I \iff j_k \leq i_k, \quad k = 1, \dots, m.$$

**Corollary 3.4.** *Let  $\widetilde{M}_i$  be the representation matrix of  $\mathcal{M}_i$  ( $i = 0, \dots, m$ ) with respect to the basis  $\{\widetilde{F}_I\}_I$ . For  $k = 1, \dots, m$ , we have*

$$\widetilde{M}_k = E_2 \otimes \dots \otimes E_2 \otimes \underset{k\text{-th}}{\widetilde{G}_k} \otimes E_2 \otimes \dots \otimes E_2, \quad \widetilde{G}_k = \begin{pmatrix} 1 & -\gamma_k^{-1} \\ 0 & \gamma_k^{-1} \end{pmatrix}.$$

$\widetilde{M}_0$  is written as

$$\widetilde{M}_0 = E_{2^m} - N_0, \quad N_0 = {}^t(\mathbf{0}, \dots, \mathbf{0}, \mathbf{v}),$$

where  $\mathbf{v} \in \mathbb{C}^{2^m}$  is a column vector whose  $I$ -th entry is

$$\begin{cases} (-1)^m \frac{(\alpha-1)(\beta-1) \prod_{k=1}^m \gamma_k}{\alpha\beta} & (I = (0, \dots, 0)), \\ (-1)^{m+|I|} \frac{(\alpha\beta+(-1)^{|I|} \prod_{k=1}^m \gamma_k) \prod_{k=1}^m \gamma_k^{1-i_k}}{\alpha\beta} & (I \neq (0, \dots, 0)). \end{cases}$$

Under the condition (1) without assuming (5),  $\widetilde{M}_0, \dots, \widetilde{M}_m$  are valid.

*Proof.* By the definition of  $\{\widetilde{F}_I\}_I$ , we have  $\widetilde{M}_i = P_m^{-1} M_i P_m$ . The first claim follows from  $Q_k^{-1} G_k Q_k = \widetilde{G}_k$  and the expressions of  $M_k$  and  $P_k$  as tensor products.

We show the second claim. Note that since all of the entries of the  $2^m$ -th column (we also say the  $(1, \dots, 1)$ -th column) of  $P_m$  are 1, we have  $P_m \mathbf{e}_{1, \dots, 1} = \mathbf{1}$  and hence  $\mathbf{e}_{1, \dots, 1}$  is an eigenvector of  $\widetilde{M}_0$  of eigenvalue  $(-1)^{m-1} \alpha^{-1} \beta^{-1} \prod_{k=1}^m \gamma_k$ , where  $\mathbf{e}_{1, \dots, 1} = {}^t(0, \dots, 0, 1) \in \mathbb{C}^{2^m}$ . By  $\widetilde{M}_0 = P_m^{-1} M_0 P_m$  and Fact 2.8,  $\widetilde{M}_0 - E_{2^m}$  should be

$$\begin{aligned} & \frac{(\beta-1)(\alpha - \prod_{k=1}^m \gamma_k)}{\alpha\beta} P_m^{-1} \cdot \mathbf{1} \cdot {}^t \mathbf{1} \cdot H P_m = \frac{(\beta-1)(\alpha - \prod_{k=1}^m \gamma_k)}{\alpha\beta} \mathbf{e}_{1, \dots, 1} {}^t \mathbf{1} H P_m \\ & = \frac{(\beta-1)(\alpha - \prod_{k=1}^m \gamma_k)}{\alpha\beta} {}^t(\mathbf{0}, \dots, \mathbf{0}, \mathbf{h}) \cdot P_m, \end{aligned}$$

where  $\mathbf{h} \in \mathbb{C}^{2^m}$  is a column vector whose  $I$ -th entry is  $H_{I,I}$ . It is sufficient to show that

$$\frac{(\beta-1)(\alpha - \prod_{k=1}^m \gamma_k)}{\alpha\beta} {}^t \mathbf{h} \cdot P_m = {}^t \mathbf{v}.$$

The  $I$ -th entry of the left-hand side is equal to

$$\begin{aligned} & \frac{(\beta-1)(\alpha - \prod_{k=1}^m \gamma_k)}{\alpha\beta} \sum_{J \leq I} \left( H_{J,J} \cdot \prod_{k=1}^m (1 - \gamma_k)^{1-i_k} \right) \\ & = \frac{(-1)^m}{\alpha\beta \prod_{k=1}^m (1 - \gamma_k)^{i_k}} \sum_{J \leq I} \left( (-1)^{|J|} \cdot \prod_{k=1}^m \gamma_k^{1-j_k} \cdot \left( \alpha - \prod_{k=1}^m \gamma_k^{j_k} \right) \left( \beta - \prod_{k=1}^m \gamma_k^{j_k} \right) \right). \end{aligned}$$

If  $I = (0, \dots, 0)$ , then this is the  $(0, \dots, 0)$ -th entry of  $\mathbf{v}$ . If we assume  $I \neq (0, \dots, 0)$ , then it equals to

$$\begin{aligned} & \frac{(-1)^m}{\alpha\beta \prod_{k=1}^m (1 - \gamma_k)^{i_k}} \sum_{J \leq I} (-1)^{|J|} \left( \alpha\beta \prod_{k=1}^m \gamma_k^{1-j_k} + \prod_{k=1}^m \gamma_k^{1+j_k} \right) \\ &= \frac{(-1)^m}{\alpha\beta \prod_{k=1}^m (1 - \gamma_k)^{i_k}} \left( \alpha\beta + (-1)^{|I|} \prod_{k=1}^m \gamma_k^{i_k} \right) \prod_{k=1}^m (\gamma_k^{1-i_k} (\gamma_k - 1)^{i_k}) \end{aligned}$$

by (7), and this coincides with the  $I$ -th entry of  $\mathbf{v}$ .  $\square$

**Lemma 3.5.**  $2^m$  vectors

$$\left( \prod_{k=1}^m \widetilde{M}_k^{i_k} \right) \cdot \mathbf{e}_{1,\dots,1} = \widetilde{M}_1^{i_1} \widetilde{M}_2^{i_2} \cdots \widetilde{M}_m^{i_m} \mathbf{e}_{1,\dots,1} \quad (I = (i_1, \dots, i_m) \in \{0, 1\}^m)$$

are linearly independent. In other words, actions  $\mathcal{M}_1, \dots, \mathcal{M}_m$  on  $f_0$  give a basis of the whole space  $\text{Sol}_{\dot{x}}$ .

*Proof.* It is sufficient to show that the  $2^m \times 2^m$  matrix

$$(\mathbf{e}_{1,\dots,1}, \widetilde{M}_1 \mathbf{e}_{1,\dots,1}, \widetilde{M}_2 \mathbf{e}_{1,\dots,1}, \widetilde{M}_1 \widetilde{M}_2 \mathbf{e}_{1,\dots,1}, \widetilde{M}_3 \mathbf{e}_{1,\dots,1}, \dots, \widetilde{M}_1 \cdots \widetilde{M}_m \mathbf{e}_{1,\dots,1})$$

is invertible. We calculate its determinant. Because of  $\widetilde{M}_k = P_m^{-1} M_k P_m$  and  $\mathbf{e}_{1,\dots,1} = P_m^{-1} \mathbf{1}$ , this matrix equals to

$$(8) \quad P_m^{-1} \cdot (\mathbf{1}, M_1 \mathbf{1}, M_2 \mathbf{1}, M_1 M_2 \mathbf{1}, M_3 \mathbf{1}, \dots, M_1 \cdots M_m \mathbf{1}).$$

By the alignment of the indices set, the right side of this product is

$$\begin{pmatrix} 1 & 1 \\ 1 & \gamma_1^{-1} \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 \\ 1 & \gamma_2^{-1} \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} 1 & 1 \\ 1 & \gamma_m^{-1} \end{pmatrix},$$

and its determinant is  $\prod_{k=1}^m (\gamma_k^{-1} - 1)^{2^{m-1}}$ . By (6), the determinant of (8) is equal to  $\prod_{k=1}^m \gamma_k^{-2^{m-1}}$ , which is not zero.  $\square$

**Proposition 3.6.** Let  $\widetilde{H} = {}^t P_m H P_m^\vee$ , which represents the intersection form  $\mathcal{I}$  with respect to the basis  $\{\widetilde{F}_I\}_I$ . Then  $\widetilde{H}$  is well-defined, and its determinant is

$$\det(\widetilde{H}) = \frac{1}{(\alpha - \prod_k \gamma_k)^{2^m} (\beta - 1)^{2^m}} \cdot \prod_{I \in \{0,1\}^m} \left( \alpha - \prod_{k=1}^m \gamma_k^{i_k} \right) \left( \beta - \prod_{k=1}^m \gamma_k^{i_k} \right).$$

In particular,  $\widetilde{H}$  is non-degenerate even in cases  $c_k \in \mathbb{Z}$  ( $k = 1, \dots, m$ ).

*Proof.* First, we show the well-definedness. For  $I = (i_1, \dots, i_m)$ ,  $I' = (i'_1, \dots, i'_m)$ , we put

$$I \cdot I' = (i_1 i'_1, \dots, i_m i'_m) \in \{0, 1\}^m.$$

Since  $H$  is diagonal, the  $(I, I')$ -entry of  $\widetilde{H}$  is

$$\begin{aligned} & \widetilde{H}_{I,I'} \\ &= \prod_{k=1}^m (1 - \gamma_k)^{1-i_k} (1 - \gamma_k^{-1})^{1-i'_k} \sum_{J \leq I, J \leq I'} \left( \prod_{k=1}^m \frac{(-1)^{j_k} \gamma_k^{1-j_k}}{\gamma_k - 1} \cdot \frac{(\alpha - \prod_{k=1}^m \gamma_k^{j_k})(\beta - \prod_{k=1}^m \gamma_k^{j_k})}{(\alpha - \prod_{k=1}^m \gamma_k)(\beta - 1)} \right) \\ &= \frac{\prod_{k=1}^m (-1)^{i_k} \gamma_k^{i_k-1} (1 - \gamma_k)^{1-i_k-i'_k}}{(\alpha - \prod_{k=1}^m \gamma_k)(\beta - 1)} \sum_{J \leq I \cdot I'} \left( (-1)^{|J|} \prod_{k=1}^m \gamma_k^{1-j_k} \cdot \left( \alpha - \prod_{k=1}^m \gamma_k^{j_k} \right) \left( \beta - \prod_{k=1}^m \gamma_k^{j_k} \right) \right). \end{aligned}$$

If  $I \cdot I' = (0, \dots, 0)$ , then  $1 - i_k - i'_k \geq 0$  ( $k = 1, \dots, m$ ), and hence

$$\tilde{H}_{I, I'} = \prod_{k=1}^m (-\gamma_k)^{i'_k} (1 - \gamma_k)^{1 - i_k - i'_k} \cdot \frac{\alpha - 1}{\alpha - \prod_{k=1}^m \gamma_k}$$

is well-defined. If  $I \cdot I' \neq (0, \dots, 0)$ , the same calculation as the proof of Corollary 3.4 shows

$$\begin{aligned} \tilde{H}_{I, I'} &= \frac{\prod_{k=1}^m (-1)^{i'_k} \gamma_k^{i'_k - 1} (1 - \gamma_k)^{1 - i_k - i'_k}}{(\alpha - \prod_{k=1}^m \gamma_k)(\beta - 1)} \left( \alpha\beta + (-1)^{|I \cdot I'|} \prod_{k=1}^m \gamma_k^{i_k i'_k} \right) \prod_{k=1}^m \left( \gamma_k^{1 - i_k i'_k} (\gamma_k - 1)^{i_k i'_k} \right) \\ &= \frac{\alpha\beta + (-1)^{|I \cdot I'|} \prod_{k=1}^m \gamma_k^{i_k i'_k}}{(\alpha - \prod_{k=1}^m \gamma_k)(\beta - 1)} \cdot \prod_{k=1}^m (-\gamma_k)^{i'_k(1 - i_k)} (1 - \gamma_k)^{(1 - i_k)(1 - i'_k)}, \end{aligned}$$

and we can see that its denominator does not vanish.

Next, we evaluate  $\det(\tilde{H})$ . Straightforward calculation and (6) show

$$\begin{aligned} \det(H) &= (-1)^{m2^{m-1}} \cdot \frac{\prod_{k=1}^m \gamma_k^{2^{m-1}} \cdot \prod_I (\alpha - \prod_{k=1}^m \gamma_k^{i_k}) (\beta - \prod_{k=1}^m \gamma_k^{i_k})}{\prod_{k=1}^m (\gamma_k - 1)^{2^m} \cdot (\alpha - \prod_{k=1}^m \gamma_k)^{2^m} (\beta - 1)^{2^m}}, \\ \det(P_m^\vee) &= \prod_{k=1}^m \left( \gamma_k^{-2^{m-1}} (\gamma_k - 1)^{2^{m-1}} \right). \end{aligned}$$

We thus have

$$\det(\tilde{H}) = \frac{1}{(\alpha - \prod_{k=1}^m \gamma_k)^{2^m} (\beta - 1)^{2^m}} \cdot \prod_{I \in \{0,1\}^m} \left( \alpha - \prod_{k=1}^m \gamma_k^{i_k} \right) \left( \beta - \prod_{k=1}^m \gamma_k^{i_k} \right),$$

and it is not zero under the condition (2).  $\square$

By this proposition, we can relax the condition to define the intersection from on  $Sol_{\dot{x}}$ .

**Corollary 3.7.** *By using  $\{\tilde{F}_I\}_I$  and  $\tilde{H}$ , the intersection form  $\mathcal{I} : Sol_{\dot{x}} \times Sol_{\dot{x}} \rightarrow \mathbb{C}(\alpha, \beta, \gamma)$  in Definition 2.5 can be extended even in cases  $c_k \in \mathbb{Z}$  ( $k = 1, \dots, m$ ).*

**Lemma 3.8.** *The eigenspace of  $\widetilde{M}_0$  with eigenvalue 1 is expressed as*

$$\ker N_0 = \ker {}^t \mathbf{v}.$$

*Proof.* This is obvious because of the expression

$$\widetilde{M}_0 = E_{2^m} - N_0 = E_{2^m} - {}^t(\mathbf{0}, \dots, \mathbf{0}, \mathbf{v}).$$

$\square$

**Remark 3.9.** If the eigenvalue  $(-1)^{m-1} \alpha^{-1} \beta^{-1} \prod_{k=1}^m \gamma_k$  of  $\widetilde{M}_0$  coincides with 1, then the  $(1, \dots, 1)$ -th entry of  $\mathbf{v}$  is zero and  $\mathbf{e}_{1, \dots, 1}$  also belongs to this eigenspace  $\ker N_0$ .

**Lemma 3.10.**

$$\ker N_0 = \{\mathbf{w} \in \mathbb{C}^{2^m} \mid {}^t \mathbf{w} \tilde{H} \mathbf{e}_{1, \dots, 1} = 0\}.$$

*Proof.* This is also obvious because of the orthogonality of the eigenspaces (Remark 2.9 (ii)) and the definition of  $\tilde{H}$ .  $\square$

## 4. IRREDUCIBILITY

We restate the main theorem and give its proof.

**Theorem 4.1.** *The monodromy representation*

$$\mathcal{M} : \pi_1(X, \dot{x}) \rightarrow GL(\text{Sol}_{\dot{x}})$$

is irreducible under the condition (1).

*Proof.* By Theorem 3.3, it is sufficient to consider the matrix representation by  $\widetilde{M}_i$  under the isomorphism  $\text{Sol}_{\dot{x}} \simeq \mathbb{C}^{2^m}$ . Let  $W \subset \mathbb{C}^{2^m}$  be an invariant subspace.

- (i) First, we suppose  $W \not\subset \ker N_0$ . We take  $\mathbf{w} \in W$  such that  $N_0 \mathbf{w} \neq \mathbf{0}$ . By the definition of  $N_0$ , the image of  $N_0$  is spanned by  $\mathbf{e}_{1,\dots,1}$ . Thus  $N_0 \mathbf{w} \neq \mathbf{0}$  implies that there exists  $\mu \neq 0$  such that  $N_0 \mathbf{w} = \mu \mathbf{e}_{1,\dots,1}$ . We obtain

$$\mathbf{e}_{1,\dots,1} = \frac{1}{\mu} N_0 \mathbf{w} = \frac{1}{\mu} (\mathbf{w} - \widetilde{M}_0 \mathbf{w}) \in W.$$

By Lemma 3.5, the  $2^m$  vectors

$$\left( \prod_{k=1}^m \widetilde{M}_k^{i_k} \right) \cdot \mathbf{e}_{1,\dots,1} \in W \quad (I = (i_1, \dots, i_m) \in \{0, 1\}^m)$$

are linearly independent. This implies  $W = \mathbb{C}^{2^m}$ .

- (ii) Next, we suppose  $W \subset \ker N_0$ . We fix an arbitrary  $\mathbf{w} \in W$ . Since  $W$  is an invariant subspace, we have

$$(\widetilde{M}_1^{i_1})^{-1} (\widetilde{M}_2^{i_2})^{-1} \dots (\widetilde{M}_m^{i_m})^{-1} \mathbf{w} \in W \subset \ker N_0$$

for any  $I = (i_1, \dots, i_m) \in \{0, 1\}^m$ . By the monodromy invariance  ${}^t \widetilde{M}_i \widetilde{H} \widetilde{M}_i^\vee = \widetilde{H}$  of the intersection matrix  $\widetilde{H}$ , commutativity between  $\widetilde{M}_1, \widetilde{M}_2, \dots, \widetilde{M}_m$ , and Lemma 3.10, we obtain

$$\begin{aligned} & {}^t \mathbf{w} \widetilde{H} (\widetilde{M}_1^{i_1} \widetilde{M}_2^{i_2} \dots \widetilde{M}_m^{i_m} \mathbf{e}_{1,\dots,1})^\vee \\ &= {}^t \mathbf{w} {}^t ((\widetilde{M}_1^{i_1})^{-1} (\widetilde{M}_2^{i_2})^{-1} \dots (\widetilde{M}_m^{i_m})^{-1}) \widetilde{H} \mathbf{e}_{1,\dots,1}^\vee \\ &= {}^t ((\widetilde{M}_1^{i_1})^{-1} (\widetilde{M}_2^{i_2})^{-1} \dots (\widetilde{M}_m^{i_m})^{-1} \mathbf{w}) \widetilde{H} \mathbf{e}_{1,\dots,1}^\vee = 0. \end{aligned}$$

The linear independence of  $\{\widetilde{M}_1^{i_1} \widetilde{M}_2^{i_2} \dots \widetilde{M}_m^{i_m} \mathbf{e}_{1,\dots,1}\}_I$  and  $\det(\widetilde{H}) \neq 0$  (Proposition 3.6) means that  $\mathbf{w} = 0$ . We thus have  $W = 0$ .

Therefore, the invariant subspaces should be the trivial ones.  $\square$

## 5. REDUCIBILITY

Recall that our irreducibility assumption (1) consists of  $2^{m+1}$  conditions for parameters. In this section, we show that if one of them is not satisfied then the monodromy representation  $\mathcal{M}$  of  $E_C(a, b, c)$  is reducible. More precisely, we have the following theorem.

**Theorem 5.1.** *Suppose that there exists  $I = (i_1, \dots, i_m) \in \{0, 1\}^m$  such that  $a^I \in \mathbb{Z}$ ,  $b^I \notin \mathbb{Z}$  or  $a^I \notin \mathbb{Z}$ ,  $b^I \in \mathbb{Z}$ . If  $a^{I'}, b^{I'} \notin \mathbb{Z}$  for any  $I' \in \{0, 1\}^m$  different from  $I$ , then the monodromy representation  $\mathcal{M}$  of  $E_C(a, b, c)$  is reducible, that is, there exists a non-trivial subspace in  $\text{Sol}_{\dot{x}}$  invariant under  $\mathcal{M}$ .*

*Proof.* We fix  $I = (i_1, \dots, i_m) \in \{0, 1\}^m$  and assume that  $a^I \in \mathbb{Z}$ ,  $b^I \notin \mathbb{Z}$ . If there exists  $j$  such that  $c_j \in \mathbb{Z}$  then

$$\begin{cases} a^{I+e_j} = a^I + (1 - c_j) \in \mathbb{Z} & \text{if } i_j = 0, \\ a^{I-e_j} = a^I - (1 - c_j) \in \mathbb{Z} & \text{if } i_j = 1, \end{cases}$$

which contradicts to the assumption. Thus we have  $c_1, \dots, c_m \notin \mathbb{Z}$  and the solutions  $F_{I'}(x)$  in (3) for  $I' \neq I$  are valid. Note that these  $2^m - 1$  solutions are linearly independent.

Hereafter, we regard  $a^I$  as an indeterminant, and consider two cases:

- (i)  $a^I$  approaches to a negative integer  $-L$ ;
- (ii)  $a^I$  approaches to a non-negative integer  $L'$ .

To prove the reducibility, we find a non-trivial invariant subspace in each case.

- (i) Note that the solution  $F_I(x)$  in (3) for this  $I$  is expressed as a non-zero constant multiple of

$$\lim_{a^I \rightarrow -L} \sin(\pi a^I) \sum_{(n'_1, \dots, n'_m)} \frac{\Gamma(a + n'_1 + \dots + n'_m) \Gamma(b + n'_1 + \dots + n'_m)}{\Gamma(c_1 + n'_1) \cdots \Gamma(c_m + n'_m) \Gamma(1 + n'_1) \cdots \Gamma(1 + n'_m)} x_1^{n'_1} \cdots x_m^{n'_m},$$

where  $n'_k$  ( $1 \leq k \leq m$ ) runs over the set

$$\begin{cases} \mathbb{N} = \{n_k \mid n_k \in \mathbb{N}\} = \{0, 1, 2, \dots\} & \text{if } k \notin I, \\ 1 - c_k + \mathbb{N} = \{1 - c_k + n_k \mid n_k \in \mathbb{N}\} & \text{if } k \in I. \end{cases}$$

Since

$$a + n'_1 + \dots + n'_m = a^I + n_1 + \dots + n_m = -L + n_1 + \dots + n_m \leq 0$$

for  $n_1 + \dots + n_m \leq L$  when  $a^I \rightarrow -L$ , some finite terms of  $\Gamma(a + n'_1 + \dots + n'_m)$  diverge. However, the poles of the Gamma function are simple, these poles are canceled by the limit of  $\sin(\pi a^I)$  as  $a^I \rightarrow -L$ . Hence the solution  $F_I(x)$  is reduced to the sum of finite terms by the limit  $a^I \rightarrow -L$ . Since  $F_I(x)$  is a polynomial times  $\prod_{k=1}^m x_k^{i_k(1-c_k)}$ , the 1-dimensional span of  $F_I(x)$  in  $Sol_x$  is invariant under  $\mathcal{M}$ . Therefore the monodromy representation  $\mathcal{M}$  is reducible in this case. Note that the representation matrices  $M_i$  ( $i = 0, 1, \dots, m$ ) in Fact 2.8 are valid under the limit and the  $I$ -th column of  $M_0$  is the  $I$ -th unit column vector of size  $2^m$ . We can also see that the 1-dimensional span of  $F_I(x)$  is invariant under  $\mathcal{M}$  by these representation matrices.

- (ii) If the parameter  $a^I$  goes to a non-negative integer  $L'$ , then the solution  $F_I(x)$  in (3) for this  $I$  reduces to the identically zero. Thus we use the fundamental system  $(\dots, F'_I(x), \dots) = (\dots, F_I, \dots)H^{-1}$ , where the diagonal matrix  $H$  is given in Fact 2.4. By the explicit form of  $H$ , we can easily see that  $F'_{I'}(x)$  for  $I' \neq I$  are valid under the limit  $a^I \rightarrow L'$ . The limit  $\lim_{a^I \rightarrow L'} F'_I(x)$  is a non-zero constant multiple of

$$\lim_{a^I \rightarrow L'} \sum_{(n'_1, \dots, n'_m)} \frac{\Gamma(a + n'_1 + \dots + n'_m) \Gamma(b + n'_1 + \dots + n'_m)}{\Gamma(c_1 + n'_1) \cdots \Gamma(c_m + n'_m) \Gamma(1 + n'_1) \cdots \Gamma(1 + n'_m)} x_1^{n'_1} \cdots x_m^{n'_m},$$

where  $n'_k$  ( $1 \leq k \leq m$ ) runs over the same set as (i). Since each term of this series converges as  $a^I \rightarrow L'$ , this limit is a solution to  $E_C(a, b, c)$  with the factor  $\prod_{k=1}^m x_k^{i_k(1-c_k)}$ . Thus the fundamental system  $(\dots, F'_I(x), \dots)$  is

valid under the limit  $a^I \rightarrow L'$ . By this change of fundamental systems, the representation matrices  $M_i$  ( $i = 0, 1, \dots, m$ ) are transformed into

$$M'_i = HM_iH^{-1}.$$

For  $k = 1, \dots, m$ , since  $M_k$  and  $H$  are diagonal, we have  $M'_k = HM_kH^{-1} = M_k = {}^tM_k$ . By Fact 2.8,  $M'_0$  is given as

$$\begin{aligned} M'_0 &= HM_0H^{-1} = H\left(E_{2^m} - \frac{(\beta-1)(\alpha - \prod_{k=1}^m \gamma_k)}{\alpha\beta} \cdot \mathbf{1} \cdot {}^t\mathbf{1} \cdot H\right)H^{-1} \\ &= E_{2^m} - \frac{(\beta-1)(\alpha - \prod_{k=1}^m \gamma_k)}{\alpha\beta} \cdot H \cdot \mathbf{1} \cdot {}^t\mathbf{1} = {}^tM_0. \end{aligned}$$

These representation matrices are valid under the limit  $a^I \rightarrow L'$ . By this limit, the  $I$ -th row of  $M'_0$  is the  $I$ -th unit row vector of size  $2^m$ . Hence the  $(2^m - 1)$ -dimensional space spanned by  $F'_{I'}$  ( $I' \neq I$ ) is invariant under  $\mathcal{M}$ .

Therefore, we obtain non-trivial invariant subspaces, and complete the proof.  $\square$

*Remark 5.2.* Even in the case of  $m = 1$ , we need detailed case analysis to give a fundamental system of solutions to  $E_C(a, b, c)$  in terms of the series (3) without the condition (1), refer to [4] and [7].

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